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An Iterated Mayer Expansion for the Yukawa Gas

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We construct a convergent expansion for the Yukawa gas at small activity and inverse temperature $\beta < 4\pi/e^2$.

KEY WORDS: Statistical mechanics; cluster expansion; renormalization group.

1. INTRODUCTION

In this paper we study a convergent expansion for the pressure and the truncated correlation functions of the two-dimensional classical Yukawa gas at low activity λ and inverse temperature $\beta < 4\pi/e^2$ (the collapse threshold).

The existence of the thermodynamic limit for the pressure, for any λ and $\beta e^2 < 4\pi$, and for the correlation functions, for any λ and $\beta e^2 < 16/\pi$, was proved in Refs. 1–3, together with the analyticity and exponential clustering of the correlation functions, for λ small and $\beta e^2 < 4\pi$. All these results were obtained using essentially Euclidean field theory techniques.

Here we propose a purely algebraic approach, which allows to prove the analyticity of the pressure and of the correlation functions around $\lambda = 0$. As usual, also the exponential clustering could be easily derived from the expansion.

The ideas used in this paper were developed in Ref. 4, where they have been applied to study the Yukawa gas in the collapse region $4\pi \leqslant \beta e^2 < 8\pi$, and found recently an interesting application to the Coulomb gas with

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fixed ultraviolet cutoff.[5] They are essentially a very natural application to these problems of the renormalization group techniques.[6] In fact the main idea is to do an expansion in each length scale separately and iteratively. This allows to carefully take into account the different strength of the interaction at different scales.

Other "iterated Mayer expansions" of the type discussed in this work are present in the recent literature.[7-9] We profited by some technical tools developed in Refs. 7, 12, and 13.

The algebraic approach of this paper can be extended to the Coulomb gas with fixed ultraviolet cutoff in the region $\beta e^2 > 8\pi$, where it is known[5] that the coefficients of the Mayer series are finite. Unfortunately, up to now, we were not able to prove the convergence of the expansion in this case.

As regards the Yukawa gas in the collapse region $4\pi \leqslant \beta e^2 < 8\pi$, it is possible to show that all the coefficients of the Mayer series are finite, except the first M, where M is the largest even integer such that $2(M-1) - \beta e^2 M/4\pi \leqslant 0$. Another open problem, whose solution is strictly related to the solution of the previous one, is the convergence of the sum of the remaining terms of the Mayer series.

2. THE MODEL

Let us consider a classical two-dimensional gas of particles of charges $\pm e$ and activity λ in a finite volume Λ, at inverse temperature β, interacting via the potential

$$C^{\leqslant N}(x-y) = \frac{1}{(2\pi)^2} \int dk\, e^{ik(x-y)} \left(\frac{1}{1+k^2} - \frac{1}{\gamma^{2N+2}+k^2} \right) \tag{2.1}$$

where γ is a scaling parameter greater than 1. As is well known, its grand canonical partition function can be written in the following way:

$$Z_\Lambda^N = \int P(d\varphi^{\leqslant N})\, e^{V_\Lambda^N} \tag{2.2}$$

where $P(d\varphi^{\leqslant N})$ is the Gaussian measure with covariance $C^{\leqslant N}$ and

$$V_\Lambda^N = \lambda \sum_{\varepsilon = \pm 1} \int_\Lambda dx\, :e^{i\bar{\alpha}\varepsilon\varphi_x^{\leqslant N}}: \tag{2.3}$$

$$\bar{\alpha} = (\beta e^2)^{1/2} \tag{2.4}$$

$$:e^{i\bar{\alpha}\varepsilon\varphi_x^{\leqslant N}}: = e^{(1/2)\bar{\alpha}^2 C_{xx}^{\leqslant N}} e^{i\bar{\alpha}\varepsilon\varphi_x^{\leqslant N}} \tag{2.5}$$

In the sequel we shall need the following standard definitions. If $f_1,..., f_n$ are random variables and $\mathscr{E}$ denotes the expectation, the truncated expectation of $f_1,..., f_n$ is defined by the equation

$$\mathscr{E}_T(f_1,..., f_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \log \mathscr{E}(e^{\sum_1^n i\lambda_i f_i})\Big|_{\lambda_1 = \cdots \lambda_n = 0} \tag{2.6}$$

If $f_1 = \cdots f_n = f$, we define also the truncated expectation of f of order n by the equation

$$\mathscr{E}_T(f; n) = \mathscr{E}_T(f,..., f) \tag{2.7}$$

Let $\mathbb{E}^{N,\Lambda}$ be the expectation with respect to the measure $(Z_\Lambda^N)^{-1} e^{V_\Lambda^N} P(d\varphi^{\leqslant N})$ and $\mathbb{E}_T^{N,\Lambda}$ the corresponding truncated expectation. The correlation functions of the gas are given by the equation[3]

$$\begin{gathered} \rho_\Lambda^N(\xi_1,..., \xi_n) = \lambda^n \mathbb{E}^{N,\Lambda} \left\{ \prod_{i=1}^n :e^{i\bar{\alpha}\varepsilon_i \varphi_{x_i}^{\leqslant N}}: \right\} \\ \xi_i = (x_i, \varepsilon_i), \qquad x_i \in \Lambda, \qquad \varepsilon_i \in \{-1, +1\} \end{gathered} \tag{2.8}$$

One has a very simple expression also for the truncated correlation functions

$$\omega_\Lambda^N(\xi_1,..., \xi_n) = \lambda^n \mathbb{E}_T^{N,\Lambda} \{ :e^{i\bar{\alpha}\varepsilon_1 \varphi_{x_1}^{\leqslant N}}:,..., :e^{i\bar{\alpha}\varepsilon_n \varphi_{x_n}^{\leqslant N}}: \} \tag{2.9}$$

This follows from Eq. (2.8) and the well-known fact that the right-hand side of Eq. (2.9) is equal to

$$\sum_{D \in \mathscr{D}(1,...,n)} (-1)^{|D|-1} (|D|-1)! \prod_{\gamma \in D} \mathbb{E}^{N,\Lambda} \left(\prod_{i \in \gamma} :e^{i\bar{\alpha}\varepsilon_i \varphi_{x_i}^{\leqslant N}}: \right) \tag{2.10}$$

where $\mathscr{D}(1,..., n)$ is the set of partitions of $(1,..., n)$ and $|D|$ denotes the number of elements in D.

For each N, $\omega_\Lambda^N(\xi_1,..., \xi_n)$ and $\rho_\Lambda^N(\xi_1,..., \xi_n)$ have a well-defined limit, as $\Lambda \nearrow \mathbb{R}^2$, $\omega^N(\xi_1,..., \xi_n)$ and $\rho^N(\xi_1,..., \xi_n)$, respectively. Also the pressure

$$p^N(\lambda) = \lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|} \log Z_\Lambda^N \tag{2.11}$$

is a meaningful expression. Furthermore all these functions are analytic around $\lambda = 0$. All these results follow from the fact that the potential $C_{xy}^{\leqslant N}$ is stable and regular.[10]

The two-dimensional Yukawa gas is here defined as the limit of this system as $N \to \infty$. In Refs. 1–3 it was shown that a similar limit (it was used a different ultraviolet cutoff) does exist if $\bar{\alpha}^2 < 4\pi$ and has some nice properties, described in the Introduction. If $\bar{\alpha}^2 \geqslant 4\pi$ the system is not stable any more, but one can show[4,11] that, if $\bar{\alpha}^2 < 8\pi$, one can recover at least the stability by doing in Eq. (2.2) the substitution

$$V_\Lambda^N \to \tilde{V}_\Lambda^N = V_\Lambda^N - \sum_1^M n \frac{1}{(2n)!} \mathscr{E}_T^N(V_\Lambda^N; 2n) \tag{2.12}$$

where $\mathscr{E}_T^N(\cdot\,; n)$ denotes the truncated expectation of order n with respect to the measure $P(d\varphi^{\leqslant N})$ and $M \geqslant 1$ depends on $\bar{\alpha}^2 (M \to \infty$ as $\bar{\alpha}^2 \to 8\pi)$.

In the sequel we shall restrict ourselves to the case

$$\bar{\alpha}^2 < 4\pi \tag{2.13}$$

The main result will be the following one:

Theorem 1. If $\bar{\alpha}^2 < 4\pi$ there exists $\lambda_0 > 0$, depending on $\bar{\alpha}^2$, such that, if $|\lambda| < \lambda_0$, $p^N(\lambda)$ converges, as $N \to \infty$, to an analytic function $p(\lambda)$.

This theorem will be proved in Sections 3 and 4. In Section 5 we shall discuss how to extend its proof in order to show the following:

Theorem 2. If $\bar{\alpha}^2 < 4\pi$ there exists $\lambda_1 > 0$, depending only on $\bar{\alpha}^2$, such that, if $|\lambda| < \lambda_1$, there exists the limit

$$\omega(\xi_1, ..., \xi_n) = \lim_{N \to \infty} \omega^N(\xi_1, ..., \xi_n) \tag{2.14}$$

3. THE EXPANSION FOR THE PRESSURE

As in Ref. 4, we start from the decomposition of the field $\varphi_x^{\leqslant N}$ as a sum of independent, identically distributed up to scale factors, Gaussian fields

$$\varphi_x^{\leqslant N} = \sum_{k=0}^N \tilde{\varphi}_x^k \tag{3.1}$$

$\tilde{\varphi}_x^k$ is, by definition, the Gaussian field with covariance

$$\tilde{C}^k(x-y) = \frac{1}{(2\pi)^2} \int dp e^{ip(x-y)} \left(\frac{1}{\gamma^{2k} + p^2} - \frac{1}{\gamma^{2(k+1)} + p^2} \right) \tag{3.2}$$

It is easy to see that

$$\tilde{C}^k(x) = \tilde{C}^0(\gamma^k x) \tag{3.3}$$

$$\tilde{C}^k(0) = \tilde{C}^0(0) = \frac{\log \gamma}{2\pi} \tag{3.4}$$

We define the "effective potential of order k" $\tilde{V}_{\Lambda}^{k}$ in the following way:

$$\tilde{V}_{\Lambda}^{k}(\varphi^{\leqslant k}) = \log \int P(d\tilde{\varphi}^{k+1})\, e^{\tilde{V}_{\Lambda}^{k+1}(\varphi^{\leqslant k+1})}, \qquad 0 \leqslant k < N \tag{3.5}$$

$$\tilde{V}_{\Lambda}^{N}(\varphi^{\leqslant N}) = V_{\Lambda}^{N} \tag{3.6}$$

This definition is such that

$$Z_{\Lambda}^{N} = \int P(d\varphi^{\leqslant k})\, e^{\tilde{V}_{\Lambda}^{k}}, \qquad 0 \leqslant k \leqslant N \tag{3.7}$$

We define also

$$\tilde{V}_{\Lambda}^{-1} = \log Z_{\Lambda}^{N} = \log \int P(d\tilde{\varphi}^{0})\, e^{\tilde{V}_{\Lambda}^{0}(\tilde{\varphi}^{0})} \tag{3.8}$$

By applying the cumulant expansion to its right-hand side, Eq. (3.5) can be written in the following way:

$$\tilde{V}_{\Lambda}^{k} = \sum_{1}^{\infty} n \frac{1}{n!} \tilde{\mathscr{E}}_{T}^{k+1}(\tilde{V}_{\Lambda}^{k+1}; n) \tag{3.9}$$

where $\tilde{\mathscr{E}}_{T}^{k}(\cdot\,; n)$ denotes the truncated expectation of order k with respect to the measure $P(d\tilde{\varphi}^{k})$.

The expansion of the effective potentials and of the pressure that we shall study is obtained by iteration of Eq. (3.9). In the first step we apply Eq. (3.9) with $k = N-1$. The step h consists in the application of Eq. (3.9) with $k = N-h$ and $\tilde{V}_{\Lambda}^{k+1}$ equal to the expansion obtained at the step $h-1$. In order to describe the structure of the iterated expansion, we need to introduce some definitions.

Definition. A 0 cluster is a single particle i. An l cluster, $l \geqslant 1$, is a family $\alpha = \{\alpha_1, \ldots, \alpha_t\}$ of $t \geqslant 2$ m clusters, $m \leqslant l-1$, which contains at least one $(l-1)$ cluster. We shall say that $\alpha_1, \ldots, \alpha_t$ are the components of α. A particle i belongs to α ($i \in \alpha$) if it belongs to one of its components. Let $S(\alpha)$ be the set of particles belonging to α and $|\alpha|$ the cardinality of $S(\alpha)$. An l custer $\alpha = \{\alpha_1, \ldots, \alpha_t\}$ has also the property that $S(\alpha_r) \cap S(\alpha_s) = \phi$ if $r \neq s$.

The 0 clusters are all equivalent, or of the same type. We shall say that two l clusters, $l \geqslant 1$, are equivalent, or of the same type, if they contain the same number of m clusters of the each type. Let C_l be the set of all l clusters and T_l the set of all types. If $\alpha \in C_l$, we shall denote by $[\alpha]$ the type of α and by $|[\alpha]|$ the number of components of α.

Given $\alpha = \{\alpha_1, ..., \alpha_l\}$ and the integer h, $0 \leqslant h \leqslant N+1$, we define a function $\sigma_\alpha^{N,l,h}(\xi_{i_1}, ..., \xi_{i_n})$, depending on the positions and on the charges $\xi_{i_r} = (x_{i_r}, \varepsilon_{i_r})$ of the particles $i_1, ..., i_n$ belonging to α, in the following way:

$$\sigma_{i_1}^{N,0,h}(\xi_{i_1}) = \lambda, \qquad h \leqslant N+1 \tag{3.10}$$

$$\sigma_\alpha^{N,l,h}(\xi_{i_1}, ..., \xi_{i_n}) = 0, \qquad \text{if } N-l+1 < h \leqslant N+1,\ l \geqslant 1$$

$$\begin{aligned}\sigma_\alpha^{N,l,h}(\xi_{i_1}, ..., \xi_{i_n}) = {} & \Bigg(\prod_{\substack{[\beta] \in T_m \\ m \leqslant l-1}} \frac{1}{N_{[\beta]}^\alpha!}\Bigg) \\ & \times \sum_h^N k \exp\Big(-\frac{\bar{\alpha}^2}{2} \sum_{r \neq s} U_{Y_r Y_S}^{\leqslant k-1}\Big) \\ & \times \Big[\exp\Big(-\frac{\bar{\alpha}^2}{2} \sum_{r \neq s} U_{Y_r Y_S}^k\Big)\Big]_T \\ & \times \Big[\prod_{r=1}^{l} \sigma_{\alpha_r}^{N,m_r,k+1}(\xi_{Y_r})\Big], \qquad h \leqslant N-l+1,\ l \geqslant 1\end{aligned} \tag{3.11}$$

where we denoted by $N_{[\beta]}^\alpha$ the number of clusters of type $[\beta]$ contained in α, by Y_r the set $S(\alpha_r)$, and by ξ_{Y_r} the coordinates of the particles in Y_r. Furthermore we used the following definitions, valid for any family $\{Y_1, ..., Y_t\}$ of mutually disjoints sets of particles:

$$U_{XY}^k = \sum_{\substack{i \in X \\ j \in Y}} \tilde{C}^k(x_i - x_j)\, \varepsilon_i \varepsilon_j \tag{3.12}$$

$$U_{XY}^{\leqslant k} = \sum_0^k {}_h\, U_{xY}^h \tag{3.13}$$

$$\Big[\exp\Big(-\frac{\bar{\alpha}^2}{2} \sum_{r \neq s} U_{Y_r Y_S}^k\Big)\Big]_T = \sum_{g \in \mathscr{G}_l^c} \prod_{(r,s) \in g} (e^{-\bar{\alpha}^2 U_{Y_r Y_S}^k} - 1) \tag{3.14}$$

In Eq. (3.14) $\mathscr{G}_n^c$ denotes the family of all connected graphs with vertices $\{1, ..., n\}$ and the pair (r, s), $1 \leqslant r < s \leqslant n$, specifies a leg of the graph g.

Remark. The l clusters here defined differ by the "trees" of Ref. 5 and the l vertices of Ref. 7 only because we sum over the "frequencies" of the components.

We claim that the iteration of Eq. (3.9) gives the following result:

$$\tilde{V}_\Lambda^k = V_\Lambda^k + \sum_2^\infty n \sum_1^{n-1} e \sum_{\substack{[k]\in T_l \\ |\alpha|=n}} \int_{[\Lambda x\{-1,+1\}]^n} d\xi_1 \cdots d\xi_n \, \sigma_\alpha^{N,l,k+1}(\xi_1,..., \xi_n)$$

$$\times \, :\exp\left(i\bar{\alpha} \sum_{i=1}^{n} \varepsilon_i \varphi_{x_i}^{\leqslant k}\right):, \qquad -1 \leqslant h \leqslant N \tag{3.15}$$

In Eq. (3.15) $\int d\xi = \sum \varepsilon \int dx$ and $\varphi_x^{\leqslant -1} \equiv 0$, $V_\Lambda^{-1} = 2\lambda\, |\Lambda|$.

Equation (3.15) can be proved very easily by induction, using Eqs. (2.3), (2.5), (3.9), and the following simple properties of the Gaussian field $\varphi^{\leqslant k}$:

$$(1) \quad :e^{i\bar{\alpha}\varepsilon\varphi_x^{\leqslant k}}: = :e^{i\bar{\alpha}\varepsilon\varphi_x^{\leqslant k-1}}: \cdot :e^{i\bar{\alpha}\varepsilon\tilde{\varphi}_x^k}: \tag{3.16}$$

(2) If $Y_1,..., Y_t$ are mutually disjoint families of particles of coordinates $\xi_{Y_i} = (x_{Y_i}, \varepsilon_{Y_i})$, $i = 1,..., t$, then

$$\mathscr{E}_T^h(:\exp(i\bar{\alpha}\varepsilon_{Y_1} \cdot \varphi_{Y_1}^{\leqslant h}):,..., :\exp(i\bar{\alpha}\varepsilon_{Y_t} \cdot \varphi_{Y_t}^{\leqslant h}):)$$

$$= :\exp\left(i\bar{\alpha} \sum_{1}^{t} {}_i \varepsilon_{Y_i} \varphi_{Y_i}^{\leqslant h-1}\right): \exp\left(-\frac{\bar{\alpha}^2}{2} \sum_{r\neq s} U_{Y_r Y_S}^{\leqslant h-1}\right)$$

$$\times \left[\exp\left(-\frac{\bar{\alpha}^2}{2} \sum_{r\neq s} U_{Y_r Y_S}^h\right)\right]_T \tag{3.17}$$

where we used the notation

$$\varepsilon_Y \cdot \varphi_Y^{\leqslant h} = \sum_{i\in Y} \varepsilon_i \varphi_{x_i}^{\leqslant h} \tag{3.18}$$

4. PROOF OF THEOREM 1

Equations (2.11), (3.8), and (3.15) imply that

$$p^N(\lambda) = 2\lambda + \sum_2^\infty n \sum_1^{n-1} e \sum_{\substack{[\alpha]\in T_e \\ |\alpha|=n}} \lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|}$$

$$\times \int_{[\Lambda x\{-1,+1\}]^n} d\xi_1 \cdots d\xi_n \, \sigma_\alpha^{N,l,0}(\xi_1,..., \xi_n) \tag{4.1}$$

The main point in the proof of Theorem 1 will be of course a bound of the series in the right-hand side of Eq. (4.1), which is uniform in N. We define

$$\|\sigma_\alpha^{N,l,h}\| = \int_{[\mathbb{R}^2 x\{-1,+1\}]^n} d\xi_1 \cdots d\xi_n \, \delta(x_i)\, e^{\bar{\alpha}^2 \bar{U}_{S(\alpha)}^{\leqslant h-1}} \, |\sigma_\alpha^{N,l,h}(\xi_1,..., \xi_n)| \tag{4.2}$$

where $i \in \alpha$ is arbitrarily choosen (the result is independent of i) and

$$\bar{U}_Y^{\leqslant h} = \frac{1}{2} \sum_{i,j \in Y} \varepsilon_i \varepsilon_j C^{\leqslant h}(x_i - x_j) \tag{4.3}$$

We can then write

$$p^N(\lambda) \leqslant 2\lambda + \sum_2^\infty n \sum_1^{n-1} e \sum_{\substack{[\alpha] \in T_l \\ |\alpha| = n}} \|\sigma_\alpha^{N,l,0}\| \tag{4.4}$$

Equation (3.11) suggests an iterative procedure for the estimation of $\|\sigma_\alpha^{N,l,h}\|$. We shall obtain the bound by using the tree formula[7,12,13] in the right-hand side of Eq. (3.14). Let us then recall some definitions.

Definition. A tree graph of order t is a mapping η from $\{2,..., t\}$ into $\{1,..., t-1\}$, such that $\eta(i) < i$. The elements of $\{1,..., t\}$ are the vertices of η. The legs of η are the couples $(\eta(i), i)$. A partial ordering on the vertices is defined by saying that i follows $\eta(i)$. To each tree we associate a function on $[0, 1]^{t-1}$

$$f_\eta(S_1,..., S_{t-1}) = \prod_{i=2}^t S_{\eta(i)} S_{\eta(i)+1} \cdots S_{i-2} \tag{4.5}$$

where empty products should be read as 1.

It is possible to show (see for example Ref. 7) that

$$\begin{aligned} \frac{1}{t!} \sum_{g \in \mathscr{G}_t^c} \prod_{(r,s) \in g} (e^{-\bar{\alpha}^2 U^k_{Y_r Y_s}} - 1) = {} & \frac{(-\bar{\alpha}^2)^{t-1}}{t} \zeta \int_{[0,1]^{t-1}} dS_1 \cdots dS_{t-1} \\ & \times \sum_\eta f_\eta(S_1,..., S_{t-1}) \left[\prod_{r=2}^t U^k_{Y_{\eta(r)} Y_r} \right] \\ & \times e^{-\bar{\alpha}^2 W^k_{Y_1 \cdots Y_t}(S_1,...,S_{t-1})} \end{aligned} \tag{4.6}$$

where ζ indicates the symmetrization with respect to $Y_1,..., Y_t$ and

$$W^k_{Y_1 \cdots Y_t}(S_1,..., S_{t-1}) = \sum_{1 \leqslant r < s \leqslant t} S_r S_{r+1} \cdots S_{s-1} U^k_{Y_r Y_s} \tag{4.7}$$

Equation (4.6) is useful because we have the following stability estimate:

$$W^k_{Y_1 \cdots Y_t}(S_1,..., S_{t-1}) \geqslant -\sum_1^t {}_r \bar{U}^k_{Y_r} \tag{4.8}$$

where $\bar{U}^k_Y$ is defined as in Eq. (4.3), with $\tilde{C}^k$ in place of $C^{\leqslant k}$.

In order to prove (4.8), we proceed as in Ref. 13. If W_Y is an expression of the form

$$W_Y = \sum_{i,j \in Y} v_{ij} \tag{4.9}$$

we define, for any $Y_1 \subset Y$.

$$W_{Y,Y_1} = \sum_{i,j \in Y_1} v_{ij} + \sum_{i,j \in Y/Y_1} v_{ij} \tag{4.10}$$

Then we define

$$\begin{aligned} \bar{W}^0_Y &= \bar{U}^k_Y, & Y &= Y_1 \cup Y_2 \cup \cdots \cup Y_t \\ \bar{W}^i_Y &= (1 - S_i)\, \bar{W}^{i-1}_{Y,Y_i \cup \cdots \cup Y_t} + S_i \bar{W}^{i-1}_Y, & i &= 1,\dots, t-1 \end{aligned} \tag{4.11}$$

It is easy to show that

$$\bar{W}^{t-1}_Y = W^k_{Y_1 \cdots Y_t}(S_1,\dots, S_{t-1}) + \sum_1^t {}_r \bar{U}^k_{Y_r} \tag{4.12}$$

Moreover, since $\bar{U}^k_Y \geqslant 0$, for any set Y of particles, Eqs. (4.10) and (4.11) easily imply, by induction, that

$$\bar{W}^i_Y \geqslant 0, \qquad i = 1,\dots, t-1 \tag{4.13}$$

Equation (4.8) follows from Eqs. (4.12) and (4.13).

Let us now observe that, if $k \geqslant h$ and $Y = \bigcup_{r=1}^t Y_r$

$$\begin{aligned} &\bar{U}^{\leqslant h-1}_Y - \frac{1}{2} \sum_{r \neq s} U^{\leqslant k-1}_{Y_r Y_s} + \sum_1^t r \bar{U}^k_{Y_r} \\ &\quad = \bar{U}^{\leqslant h-1}_Y - \bar{U}^{\leqslant k-1}_Y + \sum_1^t {}_r \bar{U}^{\leqslant k}_{Y_r} \leqslant \sum_1^t r \bar{U}^{\leqslant k}_{Y_r} \end{aligned} \tag{4.14}$$

then, if we insert Eq. (4.6) in Eq. (3.11), we obtain, using Eqs. (4.8) and (4.14),

$$\begin{aligned} e^{\bar{\alpha}^2 \bar{U}^{\leqslant h-1}_{S(\alpha)}} |\sigma^{N,l,h}_\alpha(\xi_1,\dots, \xi_n)| &\leqslant \bar{K}_\alpha \sum_h^N {}_k \left[\prod_{r=1}^t e^{\bar{\alpha}^2 \bar{U}^{\leqslant k}_{S(\alpha_r)}} |\sigma^{N,m_r,k+1}_{\alpha_r}(\xi_{Y_r})| \right] \\ &\quad \times \frac{\bar{\alpha}^{2(t-1)}}{t} \zeta \int dS_1 \cdots dS_{t-1} \sum_\eta f_\eta(S_1,\dots, S_{t-1}) \prod_{r=2}^t \\ &\quad \times \left[\sum_{\substack{i \in \alpha_{\eta(r)} \\ j \in \alpha_r}} \tilde{C}^0(\gamma^k(x_i - x_j)) \right] \end{aligned} \tag{4.15}$$

where

$$\bar{K}_\alpha = \frac{t!}{\prod_{[\beta] \in T_m, m \leqslant l-1} N^\alpha_{[\beta]}!} \tag{4.16}$$

We now insert Eq. (4.15) in Eq. (4.2) and we perform one integration for each leg of η, in an order suggested by the partial ordering of the tree, starting from the maximal vertices, and in a way depending on the choice of $i \in \alpha_{\eta(r)}$ and $j \in \alpha_r$. To be more precise, for any choice of η, $i \in \alpha_{\eta(r)}$ and $j \in \alpha_r$ [which selects one term in the right-hand side of Eq. (4.15)], we perform the integration with respect to x_j. Translation invariance and independence of definition (4.2) of i allow us to show that

$$\begin{aligned} \|\sigma_\alpha^{N,l,h}\| \leqslant{} & \bar{K}_\alpha \sum_h^N {}_k \left[\prod_{r=1}^t \|\sigma_{\alpha_r}^{N,m_r,k+1}\| \right] \frac{\bar{\alpha}^{2(t-1)}}{t} \\ & \times \left[\int_{\mathbb{R}^2} dx \tilde{C}^0(\gamma^k x) \right]^{t-1} \sum_\eta \left[\prod_{r=2}^t |\alpha_{\eta(r)}| \; |\alpha_r| \right] \\ & \times \int_{[0,1]^{t-1}} dS_1 \cdots dS_{t-1} f_\eta(S_1,..., S_{t-1}) \end{aligned} \tag{4.17}$$

The sum over η in Eq. (4.17) can be done using the tree estimate[7,12]

$$\sum_\eta \left[\prod_{r=2}^t |\alpha_{\eta(r)}| \; |\alpha_r| \right] \int dS_1 \cdots dS_{t-1} f_\eta(S_1,..., S_{t-1}) \leqslant \prod_{r=1}^{t-1} |\alpha_{r+1}| \, e^{|\alpha_r|} \tag{4.18}$$

Moreover,

$$\int_{\mathbb{R}^2} dx \tilde{C}^0(\gamma_x^k) = \gamma^{-2k} \left(1 - \frac{1}{\gamma^2}\right) \tag{4.19}$$

Then we have

$$\begin{aligned} \|\sigma_\alpha^{N,l,h}\| \leqslant{} & \bar{K}_\alpha \sum_h^N {}_k \left[\prod_{r=1}^t \|\sigma_{\alpha_r}^{N,m_r,k+1}\| \; |\alpha_r| \, e^{|\alpha_n|} \right] \\ & \times \frac{[\bar{\alpha}^2(1-1/\gamma^2)]^{t-1}}{t} \gamma^{-2k(t-1)} \end{aligned} \tag{4.20}$$

If $\bar{\alpha}^2 < 4\pi$, Eq. (4.20) easily implies by induction that

$$\|\sigma_\alpha^{N,l,h}\| \leqslant |\lambda|^{|\alpha|} \, K_\alpha^l \gamma^{-h[2(|\alpha|-1)-(\bar{\alpha}^2/4\pi)|\alpha|]} \tag{4.21}$$

where K_α^l are constants inductively defined by

$$\begin{aligned} K_i^0 &= 2 \\ K_\alpha^l &= \bar{K}_\alpha \left[\prod_{r=1}^{t} K_{\alpha_r}^{m_r} e^{2|\alpha_n|} \right] \frac{[\bar{\alpha}^2(\gamma^2-1)]^{t-1}}{t} \\ &\quad \times \sum_0^\infty {}_k \gamma^{-(k+1)[2(|\alpha|-1)-(\bar{\alpha}^2/4\pi)|\alpha|]} \end{aligned} \tag{4.22}$$

It is sufficient to observe that, by Eqs. (3.4), (4.2), (4.3),

$$\|\sigma_i^{N,0,h}\| = 2\,|\lambda|\,\gamma^{(\bar{\alpha}^2/4\pi)h} \tag{4.23}$$

and to insert Eq. (4.21) in the right-hand side of Eq. (4.20).

Equations (4.4) and (4.21) imply that

$$p^N(\lambda) \leqslant 2\,|\lambda| + \sum_2^\infty n \sum_1^{n-1} l \sum_{\substack{[\alpha]\in T_l \\ |\alpha|=n}} |\lambda|^{|\alpha|} K_\alpha^l \tag{4.24}$$

and we are left with the problem of showing that the series in the right-hand side of Eq. (4.24) is convergent, if $|\lambda|$ is small enough.

Equations (4.16) and (4.22) imply that, if $l \geqslant 1$

$$K_\alpha^l \leqslant A \frac{t!}{\prod_{[\beta]\in T_m, m\leqslant l-1} N_{[\beta]}^\alpha!} \frac{B^{t-1}}{t} \prod_{r=1}^{t} [K_{\alpha_r}^{m_r} \delta^{|\alpha_r|}] \tag{4.25}$$

where

$$\begin{aligned} A &= \gamma^2 \sum_0^\infty {}_i\gamma^{-i(2-\bar{\alpha}^2/2\pi)} \\ B &= \bar{\alpha}^2(\gamma^2-1) \\ \delta &= e^2\gamma^{-(2-\bar{\alpha}^2/4\pi)} \end{aligned} \tag{4.26}$$

Suppose γ is so large that

$$\delta \leqslant 1 \tag{4.27}$$

This is not really a restriction since $\lim_{N\to\infty} p^N(\lambda)$ is clearly independent of γ, if it exists. Then Eq. (4.25) becomes

$$K_\alpha^l \leqslant A \frac{t!}{\prod_{[\beta]\in T_m, m\leqslant l-1} N_{[\beta]}^\alpha!} \frac{B^{t-1}}{t} \prod_{r=1}^{t} K_{\alpha_r}^{m_r} \tag{4.28}$$

Let us now define

$$\psi_t^l = \sum_{\substack{[\alpha]\in T_l \\ |[\alpha]|=t}} |\lambda|^{|\alpha|} K_\alpha^l, \qquad l \geqslant 1,\ t \geqslant 2 \tag{4.29}$$

$$\xi^l = \sum_2^\infty {}_t\psi_t^l, \qquad l \geqslant 1 \tag{4.30}$$

$$\xi^0 = |\lambda|\, K_i^0 = 2\,|\lambda| \tag{4.31}$$

Then we can write Eq. (4.24) in the form

$$p^N(\lambda) \leqslant 2\,|\lambda| + \sum_1^\infty {}_l\xi^l \tag{4.32}$$

Equations (4.28) and (4.29) imply that

$$\psi_t^l \leqslant A\,\frac{B^{t-1}}{t} \sum_{\substack{[\alpha]\in T_l \\ |[\alpha]|=t}} \frac{t!}{\prod_{[\beta]\in T_m, m\leqslant l-1} N_{[\beta]}^\alpha!} \prod_{r=1}^t [K_{\alpha_r}^{m_r}\,|\lambda|^{|\alpha_r|}] \tag{4.33}$$

Since any l cluster contains at least one $(l-1)$ cluster, we have

$$\begin{aligned}\psi_t^l &\leqslant A\,\frac{B^{t-1}}{t}\left[\left(\sum_{\substack{[\alpha]\in T_m \\ m\leqslant l-1}} |\lambda|^{|\alpha|} K_\alpha^m\right)^t - \left(\sum_{\substack{[\alpha]\in T_m \\ m\leqslant l-2}} |\lambda|^{|k|} K_\alpha^m\right)^t\right] \\ &\leqslant AB^{t-1}\xi^{l-1}\left(\sum_0^{l-1} \xi^m\right)^{t-1}\end{aligned} \tag{4.34}$$

where we used the inequality

$$(a+b)^t - b^t \leqslant at(a+b)^{t-1} \tag{4.35}$$

Therefore, if $l \geqslant 1$

$$\xi^l = \sum_2^\infty {}_t\psi_t^l \leqslant A\xi^{l-1} \sum_1^\infty s\left(B\sum_0^{l-1} {}_m\xi^m\right)^S \tag{4.36}$$

Suppose λ so small that

$$4AB\xi^0 = 8\,|\lambda|\,AB \leqslant \tfrac{1}{4} \tag{4.37}$$

Then it is easy to show by induction that

$$\xi^l \leqslant \xi^0(4AB\xi^0)^l, \qquad l \geqslant 0 \tag{4.38}$$

Equations (4.32), (4.37), and (4.38) immediately imply that $p^N(\lambda)$ is uniformly bounded in N, if λ satisfies Eq. (4.37).

Theorem 1 is now a trivial consequence of dominated Lebesgue theorem. The function $p(\lambda)$ is defined like $p^N(\lambda)$ [see Eq. (4.1)] with $\sigma_\alpha^{\infty,l,0}$ in place of $\sigma_\alpha^{N,l,0}$. $\sigma_\alpha^{\infty,l,h}$ is defined inductively by Eqs. (3.10), (3.11) with $N=\infty$.

5. THE CORRELATION FUNCTIONS

The proof of Theorem 2 is essentially the same as the proof of Theorem 1. By Eqs. (2.6) and (2.9)

$$\omega_\Lambda^N(\xi_1,..., \xi_n)=\lambda^n \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \log[\tilde{Z}_\Lambda^N(\xi_1,..., \xi_n)/Z_\Lambda^N]|_{\lambda_1=\cdots=\lambda_n=0} \quad (5.1)$$

where

$$\tilde{Z}_\Lambda^N(\xi_1,..., \xi_n)=\int P(d\varphi^{\leqslant N}) \exp\left(V_\Lambda^N+\sum_1^n {}_i\lambda_i :e^{i\alpha\varepsilon_i\varphi_{x_i}^{\leqslant N}}:\right) \quad (5.2)$$

One can write an expansion for $\log \tilde{Z}_\Lambda^N(\xi_1,..., \xi_n)$ analogous to the expansion of $\log Z_\Lambda^N$ [see Eq. (3.15)]. The only difference is that now there are $n+1$ different 0 clusters, associated to the $n+1$ different terms in the exponential of Eq. (5.2).

Of course all terms present in the expansion of $\log Z_\Lambda^N$ appear also in the expansion of $\log \tilde{Z}_\Lambda^N$. Then $\log \tilde{Z}_\Lambda^N/Z_\Lambda^N$ contains only terms bounded as $|\Lambda| \to \infty$, uniformly in N, and one can very easily extend the arguments in Section 4 in order to show that $\log \tilde{Z}_\Lambda^N/Z_\Lambda^N$ and its derivatives with respect to λ, λ_1,..., λ_n converge, as $N \to \infty$, if $|\lambda|$ and $|\lambda_i|$, $i=1$,..., n, are small enough, to a limit which is analytic in λ, λ_1,..., λ_n.

Theorem 2 follows immediately from these considerations and Eq. (5.1). Also the exponential clustering is an evident property of the expansion.

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